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Some Hele Shaw flows with time-dependent free boundaries

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We consider a blob of Newtonian fluid sandwiched in the narrow gap between two plane parallel surfaces. At some initial instant, its plan-view occupies a given, simply connected domain, and its growth as further fluid is injected at a number of injection points in its interior is to be determined. It is shown that certain functionals of the domain of a purely geometric character, infinite in number, evolve in a predictable manner, and that these may be exploited in some cases of interest to yield a complete description of the motion.

By invoking images, these results may be used to solve certain problems involving the growth of a blob in a gap containing barriers. Injection at a point in a half-plane bounded by a straight line, with an initially empty gap, is shown to lead to a blob whose outline is part of an elliptic lemniscate of Booth for which there is a simple geometrical construction. Injection into a quarter-plane is also considered in some detail when conditions are such that the image domain involved is simply connected.

1. Introduction

In an earlier paper (Richardson 1972) a class of time-dependent free-boundary problems involving Hele Shaw flow was considered. In particular, the expansion of a blob of Newtonian fluid sandwiched in the narrow gap between two plane parallel surfaces as a result of the injection of further fluid into the blob at a single fixed point was examined. For blobs whose plan-view occupies a simply connected domain, it was shown that the motion possesses an infinite number of invariants which are functionals of a purely geometric character, and that these could be exploited to yield a complete analytic description of the growth of the blob in certain circumstances. Specifically, if the original domain occupied by the blob is the image of a circular disk under a conformal map by a rational function, these considerations reduce the problem to that of solving a finite system of algebraic equations.

In the present paper, we consider the growth of such a blob under the influence of injection at *several* fixed points. The analogues of the functionals introduced to examine the motion with a single injection point are no longer invariants but are found to evolve in a simple, predictable manner, and can still be exploited to yield complete analytic solutions describing the evolution of the blob. This generalization to more than one injection point is of interest because it allows one to deal with a number of problems involving the interaction of an expanding blob with barriers or boundaries within the gap. For example, consider an initial blob which is symmetric about a given line to be injected at two points symmetrically placed with respect to that line, the injection rate being the same at both points. It is evident that the growing blob will *remain* symmetric about the line, and the resultant flow in a half-plane bounded

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by this line of symmetry is the same as that which would occur with a single injection point and the boundary of the half-plane as a barrier. Similarly, the expansion of a blob confined to a quarter-plane by two semi-infinite lines meeting at right-angles under the influence of a single injection point can be treated by invoking an image system with a total of four injection points. More complex problems involving straight-line boundaries can also obviously be reduced to a consideration of a single unconfined blob with several injection points by introducing a suitable image system.

2. Basic equations

We consider the motion of a blob of fluid in the narrow gap between two parallel planes. The plan-view of the blob occupies a simply connected domain, parts of whose boundary are formed, in general, by portions of fixed boundaries representing barriers within the gap, the remaining parts of the boundary being free boundaries which advance as the blob grows. During this growth, the points where free boundaries meet fixed boundaries must be expected to move along the fixed boundaries. The motion is driven by the injection of further fluid at fixed points within the domain occupied by the blob.

Taking Cartesian co-ordinates (x, y) so that the (x, y) plane is parallel to the planes bounding the gap, standard arguments allow the problem to be reduced to one involving a spatial dependence on x and y only, these varying in the domain occupied by the plan-view of the blob. The average velocity over the gap, $\mathbf{u}$, is found to be given by

$$\mathbf{u} = \nabla\phi, \tag{2.1}$$

where ϕ, the velocity potential, is proportional to the pressure in the fluid. Incompressibility implies that this averaged velocity is divergence-free, so that

$$\nabla^2\phi = 0, \quad \text{except at the injection points.} \tag{2.2}$$

At a fixed boundary, the normal component of the averaged velocity must vanish, implying

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{at a fixed boundary.} \tag{2.3}$$

At a free boundary, we can expect a constant pressure condition to be relevant. It is easy to envisage problems in which different constant pressures are applicable on different portions of the free boundary, but we consider only the case where the same constant pressure is applicable everywhere. Since only the pressure *gradient* is relevant for the flow, we may take this constant pressure to be zero, so that we have

$$\phi = 0 \quad \text{at a free boundary.} \tag{2.4}$$

If we have injection at the points P_i for $i = 1, 2, \ldots, n$, then

$$\phi \sim \frac{Q_i}{2\pi}\log r_i \quad \text{as} \quad r_i \to 0, \tag{2.5}$$

where r_i is the distance from the point P_i, and Q_i is the rate of increase of area of the blob due to injection at P_i (that is, Q_i is the volume input rate at P_i divided by the gap width).

At a given time, with the blob occupying a given domain, conditions (2.2)–(2.5) are sufficient to define the velocity potential ϕ uniquely in that domain. The evolution in time is then determined by equating the rate of advance of the free boundary to the average velocity at the free boundary computed from this velocity potential. Thus the velocity of advance is given by

$$\mathbf{u} = \frac{\partial \phi}{\partial n}\mathbf{n} \quad \text{at a free boundary.} \tag{2.6}$$

Some comments on the relevance of the free-boundary conditions adopted here are given by Richardson (1972), and these should be borne in mind when attempting to apply this mathematical model to a given physical situation. For example, Saffman & Taylor (1958) find good agreement between their experiments and a theory using these boundary conditions in general, but the theoretical predictions differ significantly from the experimental results when the velocities involved are small.† The details of the flow near such a moving free boundary are still poorly understood and may depend on whether the boundary is advancing or retreating – and the flow near the junction of a fixed and a free boundary is obviously even more complex. An analysis bearing on the relevance of condition (2.3) at a fixed boundary is given by Thompson (1968).

It is worth noting that, at points where a fixed and a free boundary meet, the velocity of advance of the free boundary must be both perpendicular to the free boundary and parallel to the fixed boundary, according to the present mathematical model. Thus free boundaries advancing along fixed boundaries always meet them at right-angles. Condition (2.4) implies that the free boundaries are level lines of the harmonic function ϕ, while condition (2.3) implies that the fixed boundaries are level lines of any corresponding conjugate harmonic function, but this alone does not imply that they meet at right-angles unless we know also that ϕ can be continued as a harmonic function into a full neighbourhood of the point where they meet.

Suppose now that we have an arrangement which is symmetric about a given line: that is, the domain occupied by the blob and the fixed and free boundaries at some given initial time are symmetric with respect to this line, and the motion is driven by injection at points which are also symmetric with respect to this line, the injection rate at symmetrically placed points being equal for all time. The velocity potential ϕ then necessarily has equal values at symmetrically placed points and the resultant velocity of advance of the free boundaries is such as to maintain the symmetry for all time. Moreover, $\partial\phi/\partial n = 0$ on the line of symmetry so that the flow which occurs in one of the half-planes bounded by this line is the same as that which would occur if this line were replaced by a fixed straight-line boundary. Conversely, a flow taking place to one side of a straight line with this line as a fixed boundary may be treated as a symmetrical flow by invoking images. In this way, a number of problems involving fixed boundaries which are straight lines may be transformed into problems in which fixed boundaries

† Pitts (1980) obtains theoretical predictions in better agreement with observation for the Saffman–Taylor experiment with small velocities by invoking an *ad hoc* boundary condition for which no satisfactory explanation is offered, but discrepancies remain. The boundary condition used by Pitts is purely geometric in character and cannot be directly relevant for the more general circumstances of interest here.

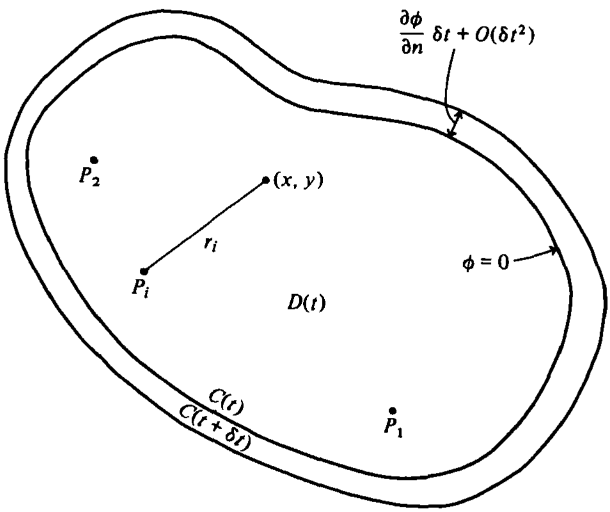

FIGURE 1. Sketch showing the expansion of the blob over a small time interval from t to $t+\delta t$.

are entirely absent. In this paper we consider only flows for which such a transformation can be effected, so that the primary difficulty is to deal with simply connected blobs bounded only by a free surface, under the influence of several injection points.

It is tempting to speculate that a similar exploitation of images is possible when fixed boundaries in the form of circular arcs are involved, but a simple generalization in this direction proves not to be possible. If we imagine a blob expanding inside a fixed circular boundary, for example, it is true that at any particular time we may determine the relevant velocity potential by introducing an image system which is the inverse of the original in the circular boundary. Unfortunately, the velocities induced by this velocity potential do not maintain the image and original systems as inverses of each other, and the problem cannot be reduced to that of the growth of a single blob with no fixed boundaries.

3. The moments

Consider a blob occupying, at time t, a simply connected domain $D(t)$ in the (x, y) plane, the entire boundary $C(t)$ of $D(t)$ being a free boundary. We have a velocity potential ϕ satisfying $\nabla^2\phi = 0$ in $D(t)$ except at the injection points P_i where ϕ is singular as indicated by relation (2.5). On the whole of $C(t)$ conditions (2.4) and (2.6) are applicable. The situation is sketched in figure 1 which shows, in particular, the expansion occurring over the small time interval from t to $t+\delta t$. Because of condition (2.6), the normal displacement taking $C(t)$ to $C(t+\delta t)$ is $(\partial\phi/\partial n)\,\delta t + O(\delta t^2)$ at each point.

Let $l(x, y)$ be any function which is harmonic in the whole (x, y) plane and does not depend on time. Define

$$L(t) = \iint_{D(t)} l(x, y)\,dx\,dy. \tag{3.1}$$

Then it is easy to see that

$$\frac{dL}{dt} \equiv \lim_{\delta t \to 0} \frac{1}{\delta t}\left\{\iint_{D(t+\delta t)} l(x,y)\,dx\,dy - \iint_{D(t)} l(x,y)\,dx\,dy\right\} = \int_{C(t)} l\frac{\partial\phi}{\partial n}\,ds. \tag{3.2}$$

If we now apply Green's theorem in the form

$$\iint_R (u\nabla^2 v - v\nabla^2 u)\,dx\,dy = \int_{\partial R}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds \tag{3.3}$$

with $u = \phi$ and $v = l$ to the region $D(t)$ with small circular disks about each injection point P_i deleted, only the line integrals round the boundary give a non-zero contribution. Because of condition (2.4), the contribution of the line integral about $C(t)$ is precisely that obtained on the right-hand side of equation (3.2). Evaluating the line integrals around the small circles about each P_i in the familiar manner by letting their radii shrink to zero and exploiting the asymptotic forms (2.5) we obtain

$$\frac{dL}{dt} = \sum_i Q_i\, l(P_i), \tag{3.4}$$

the sum being over all the injection points, where $l(P_i)$ denotes the value of $l(x,y)$ at the point P_i. We may write $Q_i \equiv Q_i(t) = dA_i(t)/dt$ where $A_i(0) = 0$, so that $A_i(t)$ is the area increase of the blob since the initial time $t = 0$ due to injection at the point P_i: that is, the total volume injected at P_i since the initial time is obtained by multiplying $A_i(t)$ by the gap width. Equation (3.4) can now be integrated to yield

$$L(t) - L(0) = \sum_i A_i(t)\, l(P_i). \tag{3.5}$$

In particular, if we choose $l(x,y) = z^n$ where $z = x+iy$ is the usual complex variable, for $n = 0, 1, 2, \ldots$, and define the *moments of the domain* by

$$M_n(t) = \iint_{D(t)} z^n\,dx\,dy \quad \text{for} \quad n = 0, 1, 2, \ldots, \tag{3.6}$$

then

$$M_n(t) = M_n(0) + \sum_i A_i(t)\, z_i^n \quad \text{for} \quad n = 0, 1, 2, \ldots, \tag{3.7}$$

where z_i is the position of the injection point P_i in the complex plane.

Given some initial domain, we may calculate $M_n(0)$. Knowing only the areas injected at each point P_i, equation (3.7) then gives $M_n(t)$ for all later time. Note, in particular, that $M_n(t)$ depends only on the total area injected at each P_i up to the time t, and not on the precise mode of injection. For example, the same result is achieved by injecting the appropriate area at each point in turn as by injecting them all simultaneously.

If we have only a single injection point at the origin, (3.7) reduces to the statement that the moments $M_n(t)$ for $n = 1, 2, 3, \ldots$ are actually constant in time, while $M_0(t)$, being the area of the domain, grows in a known manner. This special case of the general result was obtained in the earlier paper by a rather more involved argument.

The ease with which (3.7) allows the moments of the expanding domain to be calculated leads one to ask whether a simply connected domain is uniquely determined by its moments, at least for domains of some suitably restricted family. This question naturally arose in the earlier considerations of Richardson (1972), and was also raised by Aharonov & Shapiro (1976) in connection with their work on quadrature identities.

The problem was posed by Professor Shapiro during a seminar held at Kristiansand, Norway in 1975 (Bekken, Øksendal & Stray 1976, p. 193) in the form –

Let D_1 and D_2 be Jordan domains such that

$$M_n \equiv \iint_{D_1} z^n \,dx\,dy = \iint_{D_2} z^n \,dx\,dy \quad \text{for} \quad n = 0, 1, 2, \ldots . \tag{3.8}$$

Must we have $D_1 = D_2$?

It is easy to show that (3.8) cannot hold if D_1 and D_2 have disjoint closures (that is, if $\bar{D}_1 \cap \bar{D}_2$ is empty), for Runge's theorem (see, for example, Heins 1968) then implies that the analytic function on $\bar{D}_1 \cup \bar{D}_2$ which is equal to 1 on $\bar{D}_1$ and 0 on $\bar{D}_2$ can be uniformly approximated by polynomials on $\bar{D}_1 \cup \bar{D}_2$, and integration over D_1 and D_2 easily leads to a contradiction when $M_0 \neq 0$. Moreover, the result is known to be true for certain sequences of moments $M_0, M_1, M_2, \ldots$. Nevertheless, an example given by Sakai (1978) shows that the question as posed by Shapiro must be answered in the negative. In spite of this, it is clear that one must expect the domains *of some suitably restricted family* to be uniquely specified by their moments. In this paper, as in Richardson (1972), we by-pass such general questions by concentrating our efforts on a routine which enables us to *construct* domains from their moments in certain cases of interest. The success of the technique in its simplest form depends on the fact that a certain analytic function, constructed initially as a power series with the moments as its coefficients, is rational. The problem can then be reduced to one involving the solution of a system of algebraic equations by considering a particular functional equation.

It should also be remembered that, in the present problem, we seek a domain $D(t)$ which varies with time in a continuous manner. Only if it is possible to have distinct Jordan domains (say) with the same moments which are arbitrarily close to each other (in the sense of a Hausdorff metric evaluated on their closures, for example) will the moments be unable to determine the expansion of a blob in a unique manner. In determining a domain from its moments, a local uniqueness result sufficient for present purposes may well be valid even when global uniqueness fails.

4. Reduction to a functional equation

We consider the problem of determining a bounded, simply connected domain D from its moments M_n defined by (3.6). For many purposes, the fact that D and M_n are associated with a domain which varies with time will be irrelevant, and the dependence on t will often be suppressed. The methods employed are essentially those of Richardson (1972), but we introduce the ideas in a different manner.

Consider the function

$$h(x,y) = \frac{1}{\pi} \iint_D \frac{du\,dv}{z-w} \quad \text{where} \quad z = x+iy \quad \text{and} \quad w = u+iv. \tag{4.1}$$

The integral is necessarily improper when z is in D. Thus defined, $h(x,y)$ is a continuous function in the whole (x,y) plane.

If z is exterior to D, then $h(x,y)$ is equal to an analytic function of z, say $h_e(z)$. Moreover, for bounded D we may expand the integrand of (4.1) in inverse powers of z for $|z|$ sufficiently large to obtain

$$h_e(z) = \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{M_n}{z^{n+1}} \quad \text{for } z \text{ in some neighbourhood of infinity.} \tag{4.2}$$

The function $h_e(z)$ vanishes at infinity. Knowing the M_n, we may regard $h_e(z)$ as known. In general, $h_e(z)$ may be analytically continued into D, and this continuation process will produce singularities of $h_e(z)$ both in D and on the boundary, ∂D, of D. We will assume that all the singularities in fact lie in D, an assumption which is equivalent to the supposition that ∂D is an analytic curve. In the cases to be considered later, $h_e(z)$ is actually a rational function whose only singularities are poles within D.

For a point z in the interior of D, $h(x, y)$ as defined in (4.1) may be evaluated (for bounded D) by enclosing D in a circular disk, D', of sufficiently large radius and writing the integral over D as the difference of an integral over D' and an integral over $D' - D$. The integral over the circular disk D' is easily evaluated explicitly and is $\bar{z}$, where an overbar denotes the complex conjugate, while the integral over $D' - D$ is an analytic function of z for z in D. We thus have

$$h(x,y) = \begin{cases} \bar{z} + h_i(z) & \text{for } z \text{ interior to } D, \\ h_e(z) & \text{for } z \text{ exterior to } D, \end{cases} \tag{4.3}$$

where the subscripts i and e denote analyticity interior and exterior to D, respectively. In fact, relations (4.3) follow from (4.1) even when D is not bounded if the integral in (4.1) exists for all z, as shown by Aharonov & Shapiro (1976), but the expansion (4.2) is not then feasible.

The continuity of $h(x, y)$ now allows us to deduce from (4.3) a relation which must hold on the boundary ∂D. In fact,

$$h_e(z) = \bar{z} + h_i(z) \quad \text{on} \quad \partial D. \tag{4.4}$$

Let $z = f(\zeta)$ map D conformally onto $|\zeta| < 1$ in the ζ plane: we may require that $f(0) = z_0$ where z_0 is some convenient point in D, and $f'(0) > 0$, to specify $f(\zeta)$ uniquely. With ∂D an analytic curve, $f(\zeta)$ is actually analytic for $|\zeta| \leqslant 1$ and we may also contemplate its analytic continuation into $|\zeta| > 1$. The boundary relation (4.4) then transforms to a relation holding on the unit circle in the ζ plane:

$$-h_i(f(\zeta)) + h_e(f(\zeta)) = \overline{f(\zeta)} \quad \text{on} \quad |\zeta| = 1. \tag{4.5}$$

On $|\zeta| = 1$ we have $\zeta = 1/\bar{\zeta}$, so that this is equivalent to

$$-h_i(f(\zeta)) + h_e(f(\zeta)) = \overline{f(1/\bar{\zeta})}. \tag{4.6}$$

But this is now a relation between the boundary values of analytic functions, and it must therefore hold in any region into which the relevant functions may be analytically continued. If ∂D is an analytic curve we have:

(i) the first term on the left of (4.6) is analytic in $|\zeta| \leqslant 1$;

(ii) the second term on the left has singularities in $|\zeta| < 1$ whose nature is the same as that of the known singularities of $h_e(z)$ in D;

(iii) the term on the right is analytic in $|\zeta| \geqslant 1$ and tends to z_0 at infinity.

It follows that the only singularities of $\overline{f(1/\bar{\zeta})}$ are within $|\zeta| < 1$ and that their form is identical with those of $h_e(z)$. One can therefore, in general, write down the *form* of the mapping $f(\zeta)$. In particular, when $h_e(z)$ is a rational function the singularities of $\overline{f(1/\bar{\zeta})}$ are all poles, and a quantitative comparison of the singularities in (4.6) serves to fix $f(\zeta)$

completely by determining the positions and principal parts of these poles. This procedure is the same as that adopted in Richardson (1972), and is further illustrated by the two examples in the following sections. However, before considering these examples it will be helpful to insert a few remarks concerning the calculation of the moments M_n and the function $h_e(z)$ for a given domain.

Using Green's theorem, the integral over D appearing in the definition of the moments in (3.6) may be transformed to an integral over the boundary ∂D to give

$$M_n = \frac{1}{2i}\int_{\partial D} z^n \bar{z}\, dz. \tag{4.7}$$

If, now, ∂D is analytic and given by $\bar{z} = g(z)$, where $g(z)$ is analytic in a neighbourhood of ∂D, (4.7) gives M_n in a form allowing standard techniques of contour integration to be exploited. For example, a circle centre the origin, radius r, is given by $\bar{z} = r^2/z$, and (4.7) yields the corresponding moments as

$$M_n = \frac{r^2}{2i}\int_{\partial D} z^{n-1}\, dz = \begin{cases} \pi r^2 & \text{for} \quad n = 0, \\ 0 & \text{for} \quad n = 1, 2, 3, \ldots. \end{cases}$$

Equation (4.7) is also useful when ∂D is piece-wise analytic – when it is a polygon, for example.

If the whole of ∂D is given by $\bar{z} = g(z)$, as above, and we can write $g(z) = g_e(z) + g_i(z)$ where $g_e(z)$ is analytic exterior to D and vanishes at infinity, while $g_i(z)$ is analytic interior to D, then $h_e(z) = g_e(z)$ (see Richardson 1972), thus making the preliminary calculation of the moments unnecessary. For the circle of radius r centre the origin we therefore have $h_e(z) = r^2/z$.

If a domain is translated through a distance represented by the complex number a, then evidently

$$M_n \to \sum_{k=0}^{n} \binom{n}{k} a^k M_{n-k}, \quad \text{where} \quad \binom{n}{k}$$

is the usual binomial coefficient, and $h_e(z) \to h_e(z-a)$. Thus, for the circle of radius r centred at $z = a$ we have $M_n = \pi r^2 a^n$ for $n = 0, 1, 2, \ldots$ and $h_e(z) = r^2/(z-a)$, results which follow equally easily from the earlier remarks and the fact that such a circle is given by $\bar{z} = \bar{a} + r^2/(z-a)$.

It should also be borne in mind that both M_n and $h_e(z)$ are additive domain functionals for disjoint domains, a property which will prove useful in the examples to follow.

When we wish to emphasize that we are dealing with a domain $D(t)$ which is varying with time, we write the function $h_e(z)$ derived from the domain $D(t)$ at time t as $h_e(z; t)$. (3.7) and (4.2) together then imply that

$$h_e(z; t) = h_e(z; 0) + \frac{1}{\pi}\sum_i \frac{A_i(t)}{z - z_i}. \tag{4.8}$$

This relation generally allows one to avoid an explicit calculation of the moments, though their values are often of independent interest.

5. Injection into a half-plane

We consider injection at the single point $z = 1$ in the half-plane $x > 0$ bounded by the infinite straight-line barrier $x = 0$. Beginning with an empty gap, we expect the blob to grow initially as a circular disk centred at the injection point, and it is easy to confirm that the procedure given in the previous section leads to this conclusion.† Indeed, translating the origin to the injection point temporarily, it is equivalent to the statement that a simply connected domain with $M_n = 0$ for $n = 1, 2, 3, \ldots$ must be a circular disk centred on the origin, and this result has been proved under more general conditions than are required in present circumstances. (See Aharonov & Shapiro 1976 for a discussion bearing on this aspect of the problem, and references to earlier work.)

With injection at the point $z = 1$, the blob will grow as a circular disk centred on $z = 1$ until an area π has been injected, when the disk just touches the boundary $x = 0$ at the origin. To follow the further expansion of the blob as it runs along this boundary, we invoke images so that we have two injection points at $z = \pm 1$ with equal injection rates and a situation which is symmetric about the line $x = 0$. The initial state for this image system when the blob first touches the boundary $x = 0$ (taken as the time origin $t = 0$ for present purposes) consists of the union of two circular disks of unit radius centred at the points $z = \pm 1$ and for this we have

$$M_n(0) = \begin{cases} 2\pi, & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases}$$

and

$$h_e(z; 0) = \frac{1}{z-1} + \frac{1}{z+1}.$$

From (4.8), it follows that when the area of the blob in $x > 0$ is $A > \pi$, that is when a further area $A - \pi$ has been injected at both $z = \pm 1$ since time $t = 0$, we have

$$h_e(z; t) = \frac{A}{\pi}\left(\frac{1}{z-1} + \frac{1}{z+1}\right). \tag{5.1}$$

This same form for $h_e(z)$ is, of course, valid for the original blob plus its image for all $A > 0$, giving two disjoint circular disks for $0 < A < \pi$, if we generalize the definitions in an obvious manner.

In fact, it will prove more convenient to work with an equivalent radius, R, for the blob, defined by

$$A = \pi R^2, \tag{5.2}$$

so that (5.1) becomes

$$h_e(z; t) = R^2\left(\frac{1}{z-1} + \frac{1}{z+1}\right). \tag{5.3}$$

Since $h_e(z; t)$ has two simple poles, it follows from (4.6) that the function $z = f(\zeta)$ mapping the domain in the z plane at time t to the interior of the unit circle in the ζ plane is such that $\overline{f(1/\bar{\zeta})}$ has two simple poles in $|\zeta| < 1$. Moreover, if we impose $f(0) = 0$ and $f'(0) > 0$ as the extra conditions required to determine $f(\zeta)$ uniquely,

† It is a limiting case of the example in Richardson (1972).

we have $\overline{f(1/\bar{\zeta})} \to 0$ as $|\zeta| \to \infty$. Exploiting the fact that we expect the domain in the z plane to be symmetric about both the lines $x = 0$ and $y = 0$,† implying

$$f(\zeta) = \overline{f(\bar{\zeta})} = -f(-\zeta),$$

we can take the poles of $\overline{f(1/\bar{\zeta})}$ to be at the points $\pm\zeta_0$ with

$$\overline{f(1/\bar{\zeta})} = \frac{\alpha\zeta}{\zeta^2-\zeta_0^2}, \tag{5.4}$$

implying

$$f(\zeta) = \frac{\alpha\zeta}{1-\zeta_0^2\zeta^2}, \tag{5.5}$$

where both α and ζ_0 are real, $\alpha > 0$ and $0 < \zeta_0 < 1$.

Near $\zeta = \zeta_0$, corresponding to $z = 1$, (5.3) implies that

$$\begin{aligned} h_e(f(\zeta)) &= \frac{R^2}{f(\zeta)-1} + \text{regular terms} \\ &= \frac{R^2}{f(\zeta_0)-1+f'(\zeta_0)(\zeta-\zeta_0)} + \text{regular terms.} \end{aligned}$$

Comparing the positions and residues of the poles on both sides of (4.6) we thus require $f(\zeta_0) = 1$ and $\alpha f'(\zeta_0) = 2R^2$. With $f(\zeta)$ given by (5.5), these two conditions lead to

$$\alpha\zeta_0 = 1-\zeta_0^4 \quad \text{and} \quad \zeta_0^4 - 2R^2\zeta_0^2 + 1 = 0. \tag{5.6}$$

Since we require $0 < \zeta_0 < 1$ for $R > 1$, we thus have

$$\zeta_0 = [R^2-(R^4-1)^{\frac{1}{2}}]^{\frac{1}{2}} \quad \text{and} \quad \alpha = (1-\zeta_0^4)/\zeta_0. \tag{5.7}$$

With these values of α and ζ_0, the function $f(\zeta)$ in (5.5) gives the required mapping when the blob has area $A = \pi R^2$. The mapping (5.5) may be written as

$$f(\zeta) = \frac{\alpha}{\zeta^{-1}-\zeta_0^2\zeta}, \tag{5.8}$$

which exhibits it as a Joukowski mapping followed by an inversion. The boundary of the blob is thus a curve formed by inverting an ellipse with respect to its centre. Such a curve has been called an elliptic lemniscate of Booth by Loria (1902), recognizing its appearance in the work of the Reverend James Booth (1873, 1877). The standard Cartesian equation of the elliptic lemniscate of Booth is

$$(x^2+y^2)^2 = a^2x^2+b^2y^2, \tag{5.9}$$

the isolated point at the origin which satisfies this relation not being regarded as part of the curve. In this form, the curve is the inverse of the ellipse $a^2x^2+b^2y^2 = 1$ in the unit circle, centre the origin. It is also the locus of the foot of the perpendicular from the origin to a variable tangent of the ellipse $(x/a)^2+(y/b)^2 = 1$; that is, it is the pedal curve of this ellipse with respect to the origin. To obtain the required outline of the blob in the form (5.9) we must take

$$a^2 = 2(R^2+1), \quad b^2 = 2(R^2-1). \tag{5.10}$$

† It is not, in fact, *necessary* to invoke symmetry at this stage. The pole positions and residues may be put as four arbitrary complex numbers subject only to the requirements that the poles of $\overline{f(1/\bar{\zeta})}$ lie in $|\zeta| < 1$, $f(0) = 0$ and $f'(0) > 0$. The symmetry then emerges naturally in the subsequent solution, but we here anticipate the symmetry to simplify the algebra of the presentation.

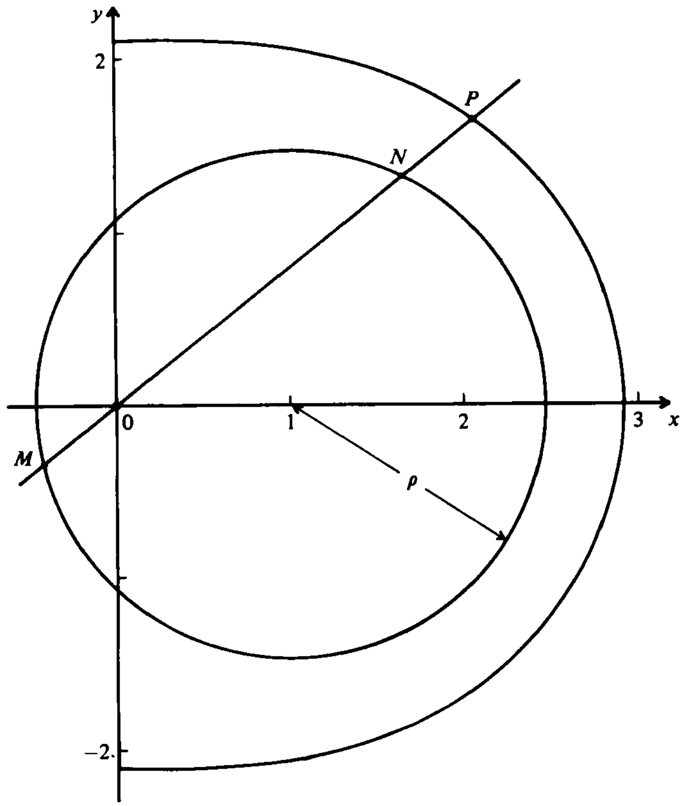

FIGURE 2. Construction of the elliptic lemniscate of Booth giving the blob outline. For a blob of area $A = \pi R^2$ take $\rho^2 = \frac{1}{2}(R^2+1)$. $OP = MN$ and the locus of P is the required outline.

A further construction for the lemniscate is also of interest, and this is illustrated in figure 2. Adapting magnitudes to the present problem, draw a circle of radius $\rho > 1$ centred on the injection point at $(x, y) = (1, 0)$: to obtain the outline for a blob of area $A = \pi R^2$ we need to take $\rho^2 = \frac{1}{2}(R^2+1)$. If a variable line through the origin, O, cuts the circle in M and N, take P on this line so that $OP = MN$. The locus of P is the required outline. This construction may be interpreted as a law of reflection for the expanding blob, though the areas of the blob and the disk used in its construction are not equal.

The outline cuts the x axis at $x = \{2(R^2+1)\}^{\frac{1}{2}}$ and meets the y axis at

$$y = \pm\{2(R^2-1)\}^{\frac{1}{2}}.$$

For $1 < R^2 < 3$ the blob is non-convex, its maximum width in the y direction being R^2+1 attained at $x = \frac{1}{2}\{(R^2+1)(3-R^2)\}^{\frac{1}{2}}$. When $R^2 = 3$ the maximum width in the y direction is at $x = 0$, this maximum width being 4 as the points of intersection with the y axis are then at ± 2. For $R^2 \geqslant 3$ the blob is convex, and tends to a semi-circle with diameter on $x = 0$ as $R \to \infty$.

A consideration of the limit $R \to 1+$ is of interest mathematically, for then $\alpha \to 0+$ and $\zeta_0 \to 1-$ in such a way that the lemniscate tends to the initial state with two circles of unit radius touching tangentially. Nevertheless, $f(\zeta)$ tends to the zero constant for $|\zeta| < 1$ as is required by the Carathéodory kernel theorem (see, for example, Carathéodory 1952, p. 74 *et seq.*). This initial state is the inverse of the lines $x = \pm\frac{1}{2}$ in the unit circle, and also the locus of points at which lines from the origin and either of the points $(\pm 2, 0)$ subtend a right-angle, these being the forms to which the two constructions of the lemniscate of Booth from an ellipse mentioned earlier degenerate; the construction of figure 2 degenerates to the case $\rho = 1$.

6. Injection into a quarter-plane

We consider injection at the single point $z = 1+iY$ in the quadrant $x > 0, y > 0$ bounded by the straight-line barriers $x = 0$ and $y = 0$. By symmetry, we may confine attention to the range $Y \geqslant 1$. Beginning with an empty gap, the initial growth follows that already determined in § 5; the situation is merely translated a distance Y in the y direction.

If $Y \geqslant 2$, the blob grows as a circular disk until it hits the line $x = 0$ and then spreads along this boundary until, when its area is $A_0 = \pi R_0^2$ where $R_0^2 = 1+\frac{1}{2}Y^2$, it first encounters the boundary $y = 0$ at the origin, the analysis of the previous section describing this phase of the motion in detail. To follow the motion beyond this point we invoke an image system which is symmetric about both $x = 0$ and $y = 0$, and which has four injection points at $\pm 1 \pm iY$, the injection rate being the same at all four points.

If $Y = 1$, it is evident that the initial circular blob hits both boundaries $x = 0$ and $y = 0$ simultaneously as it grows. If $1 < Y < 2$, the line $x = 0$ is encountered first, but the blob then hits the line $y = 0$ at a point away from the origin. In either case, the subsequent flow may again be treated by invoking images, but a doubly connected image domain is involved. Since the present work considers only simply connected domains, we here restrict attention to the range $Y \geqslant 2$.

When the blob has an area $A = \pi R^2$ with $R > R_0$ for $Y \geqslant 2$, the domain occupied by the whole image system has

$$h_e(z) = R^2\left(\frac{1}{z-1-iY}+\frac{1}{z-1+iY}+\frac{1}{z+1-iY}+\frac{1}{z+1+iY}\right). \tag{6.1}$$

This follows from (4.8), (5.3), the fact that $h_e(z)$ is an additive domain functional for disjoint domains, and from its known behaviour under translation.

Equation (4.6) now implies that the required mapping $z = f(\zeta)$ is such that $\overline{f(1/\bar{\zeta})}$ has four simple poles in $|\zeta| < 1$; taking $f(0) = 0$ and $f'(0) > 0$ to determine $f(\zeta)$ uniquely again implies that $\overline{f(1/\bar{\zeta})}$ vanishes at infinity. Exploiting the symmetry, we can take the poles at $\pm\zeta_0$ and $\pm\bar{\zeta}_0$, where $0 < \arg\zeta_0 < \frac{1}{2}\pi$, say, and

$$\overline{f(1/\bar{\zeta})} = \frac{\alpha}{\zeta-\zeta_0}+\frac{\alpha}{\zeta+\zeta_0}+\frac{\bar{\alpha}}{\zeta-\bar{\zeta}_0}+\frac{\bar{\alpha}}{\zeta+\bar{\zeta}_0}, \tag{6.2}$$

whence

$$f(\zeta) = \frac{\bar{\alpha}\zeta}{1-\bar{\zeta}_0\zeta}+\frac{\bar{\alpha}\zeta}{1+\bar{\zeta}_0\zeta}+\frac{\alpha\zeta}{1-\zeta_0\zeta}+\frac{\alpha\zeta}{1+\zeta_0\zeta}, \tag{6.3}$$

where $\mathrm{Re}\{\alpha\} > 0$ is necessary for $f'(0) > 0$.

Near $\zeta = \zeta_0$ corresponding to $z = 1+iY$ we have, from (6.1)

$$h_e(f(\zeta)) = \frac{R^2}{f(\zeta)-1-iY} + \text{regular terms}$$
$$= \frac{R^2}{f(\zeta_0)-1-iY+f'(\zeta_0)(\zeta-\zeta_0)} + \text{regular terms}.$$

Balancing the poles in (4.6) thus requires $f(\zeta_0) = 1+iY$ and $\alpha f'(\zeta_0) = R^2$. With $f(\zeta)$ given by (6.3), these conditions are

$$2\zeta_0\left(\frac{\bar{\alpha}}{1-|\zeta_0|^4}+\frac{\alpha}{1-\zeta_0^4}\right) = 1+iY \tag{6.4}$$

and

$$\frac{|\alpha|^2(1+|\zeta_0|^4)}{(1-|\zeta_0|^4)^2}+\frac{\alpha^2(1+\zeta_0^4)}{(1-\zeta_0^4)^2} = \tfrac{1}{2}R^2. \tag{6.5}$$

For given $Y \geqslant 2$ and $R > R_0$ these are to be solved for ζ_0 and α.

A formal asymptotic expansion for large R shows that

$$\zeta_0 \sim \frac{1+iY}{2R}\left\{1+\left[\frac{1}{32}(Y^2-1)^2+\frac{iY}{8}(Y^2-1)\right]R^{-4}+O(R^{-8})\right\} \quad \text{as} \quad R\to\infty, \tag{6.6}$$

while

$$\alpha \sim \tfrac{1}{2}R\left\{1+\left[-\frac{3}{32}(Y^2-1)^2+\frac{3iY}{8}(Y^2-1)\right]R^{-4}+O(R^{-8})\right\} \quad \text{as} \quad R\to\infty. \tag{6.7}$$

In particular, $f(\zeta) \sim 2R\zeta$ as $R\to\infty$ for $|\zeta| \leqslant 1$, so that the blob tends, as expected, to a circular form, the complete image approaching a circular disk of area $4\pi R^2$.

A formal asymptotic expansion for the solution of (6.4) and (6.5) when $R\to R_0+$ is best obtained by first noting that then $\zeta_0 \to i$. Writing

$$\zeta_0 = i+\delta e^{i\theta}, \tag{6.8}$$

where δ is real and small, and θ is real, we may expand θ, α and R^2 in powers of δ to obtain

$$\theta = \theta_0-\frac{\delta}{(Y^2+4)^{\frac{1}{2}}}+O(\delta^2) \tag{6.9}$$

and

$$\alpha = iY\delta e^{i\theta_0}\left\{1-\frac{Y+4i}{2(Y^2+4)^{\frac{1}{2}}}\delta+O(\delta^2)\right\}, \tag{6.10}$$

where

$$\tan\theta_0 = -\tfrac{1}{2}Y \quad \text{with} \quad -\tfrac{1}{2}\pi < \theta_0 < 0, \tag{6.11}$$

and

$$R^2 = R_0^2+\frac{Y^2(Y^2-4)}{Y^2+4}\delta^2+\frac{Y^3(Y^2-4)}{(Y^2+4)^{\frac{3}{2}}}\delta^3 + \frac{30Y^8+797Y^6-819Y^4-1780Y^2+160}{24(Y^2+4)^3}\delta^4+O(\delta^5). \tag{6.12}$$

From (6.12) it follows that δ is of order $(R-R_0)^{\frac{1}{2}}$ for $Y > 2$, but of order $(R-R_0)^{\frac{1}{4}}$ for $Y = 2$.

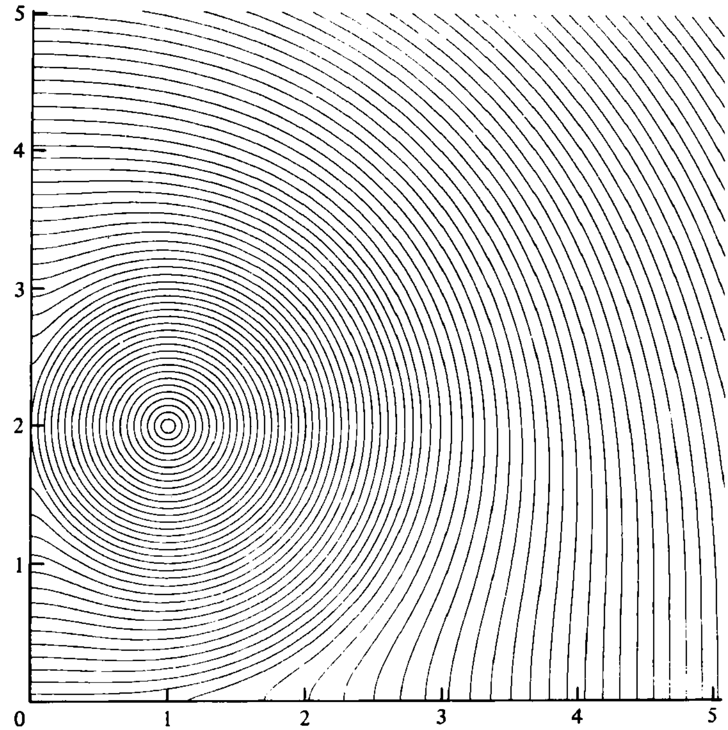

FIGURE 3. Growth of a blob in a quarter-plane. The injection point is at (1, 2). With the area of the blob as $A = \pi R^2$, the outlines are plotted at increments of 0·05 in R.

These asymptotic forms for $R \to \infty$ and $R \to R_0+$ must be complemented by a numerical solution for intermediate values of R. For this purpose, it seems simplest first to solve (6.4) for α as a function of ζ_0 by taking its conjugate and eliminating $\bar{\alpha}$, and then to use this to substitute for α in (6.5), thus obtaining a single equation to be solved for ζ_0 when Y and R are given. Rewriting this as $\sigma(\zeta_0) + i\tau(\zeta_0) = 0$ where σ and τ are *real* functions of ζ_0, a standard search routine to find the value of ζ_0 giving the minimum of $\sigma^2 + \tau^2$ proved successful in all cases computed, the proximity of this minimum to zero when compared with the value of $\sigma^2 + \tau^2$ for neighbouring values of ζ_0 giving an effective estimate for the accuracy of the solution obtained.

To plot the outlines for a particular expanding blob with a given $Y \geqslant 2$ it is simplest to begin with the largest value of R of interest and to use the dominant term of (6.6) as a preliminary estimate of the required root ζ_0 in the search routine. This term alone actually gives a fairly accurate value for ζ_0 even for values of R as low as $2R_0$, as might be expected from the fact that the expansion (6.6) proceeds in inverse *fourth* powers of R. For successively smaller values of R, previously determined roots may be used to furnish a preliminary estimate instead.

Figure 3 shows the growth of a blob in the quarter-plane with the injection point at $z = 1 + 2i$. The outline evolves from its initial circular shape, through a family of elliptic lemniscates of Booth as determined in § 5, to a form described by the analysis of this section.

7. Concluding remarks

It is evident that the techniques of the present paper can be used to solve other problems involving the growth of a simply connected blob under the influence of several injection points, whether these are obtained by invoking images or not. It is equally evident that, even when the initial blob is simply connected, the use of images can lead to the consideration of domains which are no longer simply connected, as with the expansion of a blob in a quarter-plane when $1 \leqslant Y < 2$, as mentioned in § 6. In these circumstances, the moments defined in (3.6) alone can no longer suffice to determine the domain: for example, any circular annulus centred on the origin has $M_n = 0$ for $n = 1, 2, 3, \ldots$, M_0 being its area. However, for multiply connected domains it is possible to define an *extended* system of 'moments' which behave in a predictable manner during injection, and which *do* seem sufficient to determine the domain, at least in some cases of interest. The details have yet to be worked out, but the principles are simple extensions of those in the present paper.

The use of images can also lead to a consideration of unbounded domains, even when the original domain is bounded: injection into an infinite strip obviously furnishes a problem of this kind. The moments of an unbounded domain can only be defined for a restricted class of such domains, but an unbounded domain can always be treated as the limit of a bounded one for present purposes, just as the expansion of a blob which initially occupies a half-plane was treated as a limiting case of an initially circular blob in Richardson (1972).

In other circumstances (for example, injection inside a rectangular boundary) the use of images leads one to consider unbounded domains of infinite connectivity, thus compounding the difficulties in the previous two paragraphs. Though a solution of such problems may be feasible, it seems likely that a direct consideration of the original situation without invoking images may be more profitable. In any case, images can only be used for very special geometries so that a technique for handling interactions with boundaries which does not exploit them is highly desirable.

The analysis of § 3 concerning the moments and their predictable behaviour governed by equation (3.7) can be generalized in a number of directions. If we consider a three-dimensional blob of fluid in a homogeneous porous medium expanding under the influence of injection points within it (or contracting because of suction), a constant pressure condition being relevant at the free surface, the standard mathematical model is the three-dimensional analogue of that obtained for the Hele Shaw flow, Laplace's equation still being relevant. One finds that the 'generalized moments' formed by integrating solid harmonics over the domain now behave predictably. If we have a two-dimensional problem involving injection into a non-constant, but slowly varying, gap, or a three-dimensional problem involving a blob of fluid in a porous medium whose permeability depends on position, the relevant equation is no longer that of Laplace. Nevertheless, one finds that 'generalized moments', defined by integrating a suitable set of solutions of this equation over the domain occupied by the fluid, again satisfy an equation similar to (3.7). This fact can obviously be used as a check on any numerical solution obtained to a problem of this kind, but whether these generalized moments can be exploited more directly to construct non-trivial explicit solutions in these circumstances remains to be seen.

The numerical computations involved in the solution of equations (6.4) and (6.5) and the production of figure 3 were performed by Mr N. K. Mooljee of the Edinburgh Regional Computing Centre. His valuable assistance is gratefully acknowledged.

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